SOLVING SYSTEMS OF 2-D LINEAR EQUATIONS **MATH1314/GWJ/08**

Is there a solution?

You won't know for sure until you try, but there are some things to watch for, if you know how to "read the equation" for its coefficients. Consider:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

if: $a_1/a_2 = b_1/b_2$ but not $= c_1/c_2$ then these are likely to be parallel lines "no solution", "inconsistent system"

if: $a_1/a_2 = b_1/b_2 = c_1/c_2$ the two equations are the same line on the graph "infinitely-many solutions", "dependent system"

if: a_1/a_2 is not $= b_1/b_2$ then there is one point where the lines cross (there is one "unique" solution)

Finding that solution (if there is one)

graphical, with an appropriate calculator
two algebraic methods
three matrix methods

Algebraic / substitution:

(1) look for one variable already or easily isolated
(2) plug the expression for the isolated variable back into the other equation & solve for the value of the remaining variable
(3) plug that value back into a two-variable equation and solve for the value of the other variable

example:

$4x + 3y = 10$	
$2x + y = 4$	isolate $y = -2x + 4$ & plug that into other one
$4x + 3(-2x + 4) = 10$	now distribute the +3 onto the ( )
$4x - 6x + 12 = 10$	now combine like terms
$-2x = -2$	divide off the coefficient of x
$x = 1$	now use this in our eqn for y above
$y = -2(1) + 4$	simplify
$y = -2 + 4 = 2$	and check your answer

$4x + 3y = 4(1) + 3(2) = 4 + 6 = 10$ VERIFIED

$2x + y = 2(1) + 2 = 2 + 2 = 4$ VERIFIED

Algebraic / elimination (addition): (1) look for “natural” (variable with equal & opposite coefficients) and add, or “almost-a-natural” (variable with equal coefficients) and subtract
(2) failing (1), determine a multiplier on one eqn that gets you a “natural” and apply it
(3) failing (2) determine two multipliers that get you a natural and apply them
(4) add (or subtract) to eliminate selected variable
(5) solve the result for the value of the remaining variable
(6) substitute that value into any of the two-variable eqns to find the other value

same example: $4x + 3y = 10$ no “natural” or “almost-a-natural”, but ….
$2x + y = 4$ can use single -2 multiplier on second eqn

$4x + 3y = 10$
$\underline{-4x - 2y = -8}$ adding eqns eliminates x, plug y value into
$y = 2$ any of the two-variable eqns to find x

$2x + y = 4$ <<< we use this one
$2x + (2) = 4$ <<< plug y = 2 in
$2x = 2$ <<< subtract 2 from both sides
$x=1$ <<< divide off the coefficient of 2

so x=1, y=2 (same solution as above)

Matrix / row operations: (1) set up the augmented matrix
(2) add multiples of rows to multiples of rows until you have a diagonal of 1’s and zeroes elsewhere
(3) the last column is your solution

same example: $4x + 3y = 10$ becomes $\begin{vmatrix} 4 & 3 & 10 \\ 2 & 1 & 4 \end{vmatrix}$ (it’s the 2 & 3 that must become 0’s)
$2x + y = 4$

subtract triple the second row from the first (second row unchanged)

$$\begin{matrix} -2 & 0 & -2 \\ 2 & 1 & 4 \end{matrix}$$

add the top row to the second row (top row unchanged)

$$\begin{matrix} -2 & 0 & -2 \\ 0 & 1 & 2 \end{matrix}$$

now divide the top row by -2

$$\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \quad \begin{array}{l} \text{<<< that's x} \\ \text{<<< that's y} \end{array}$$

2x2 determinant of a square matrix: find the product of the main diagonal and subtract from it the product of the off-diagonal

(main diagonal: up-left to low-right)

Example: $\det \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4(1) - 2(3) = 4 - 6 = -2$

Matrix / Cramer's Rule:

(1) set up the augmented matrix
(2) find determinant of matrix of coefficients
(3) substitute the constants column for the x column and compute that determinant; that value divided by the coeff. determinant is x
(4) substitute the constants column for the y column and compute that determinant; that value divided by the coeff. determinant is y

same example: $4x + 3y = 10$, $2x + y = 4$ becomes

$$\begin{array}{ccc} 4 & 3 & 10 \\ 2 & 1 & 4 \\ \wedge & \wedge & \wedge \\ \text{x's} & \text{y's} & \text{const} \end{array} \text{ (column associations)}$$

det|coefficients| = $\det \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4(1) - 2(3) = -2$ (same as above)

$$\text{x-det} = \det \begin{vmatrix} \overset{\text{sub}}{10} & 3 \\ 4 & 1 \end{vmatrix} = 10(1) - 4(3) = 10 - 12 = -2$$

$$\text{y-det} = \det \begin{vmatrix} 4 & \overset{\text{sub}}{10} \\ 2 & 4 \end{vmatrix} = 4(4) - 2(10) = 16 - 20 = -4$$

x = x-det/det(coeff) = -2/(-2) = 1
y = y-det/det(coeff) = -4/(-2) = 2

so, x = 1, y = 2, same as above

Matrix Inversion Method:

(1) form matrix of coefficients and vector of constants
(2) invert the matrix of coefficients
(3) premultiply the inverted matrix of coefficients onto the vector of constants; the resulting vector is the values of the variables

same example: $4x + 3y = 10$ becomes $\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 10 \\ 4 \end{vmatrix}$

$2x + y = 4$

To invert a 2x2, swap elements on the main diagonal, and make off-diagonal elements negative; then divide that matrix by the determinant of the same matrix

$$\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}^{-1} = \frac{\begin{vmatrix} 1 & -3 \\ -2 & 4 \end{vmatrix}}{\text{Det}\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} 1 & -3 \\ -2 & 4 \end{vmatrix}}{-2} = \begin{vmatrix} -1/2 & 3/2 \\ 1 & -2 \end{vmatrix}$$

$$\text{and } \begin{vmatrix} x \\ y \end{vmatrix} = \underbrace{\begin{vmatrix} -1/2 & 3/2 \\ 1 & -2 \end{vmatrix}}_{\text{inverse}} \underbrace{\begin{vmatrix} 10 \\ 4 \end{vmatrix}}_{\text{const}} = \underbrace{\begin{vmatrix} (-10/2 + 12/2) \\ (10 - 8) \end{vmatrix}}_{\text{simplify}} = \begin{vmatrix} 1 \\ 2 \end{vmatrix}$$

or $x = 1$, $y = 2$ (same as before)

Calculator graphing:

(1) put both equations in $y = mx + b$ format
(2) enter both equations into the calculator
(3) graph the pair of equations
(4) run the appropriate command to find the coordinates of the crossing point

(details depend upon the specific calculator)

REMEMBER – always check your work!!!!